DEPARTMENT OF APPLIED MATHEMATICS,
UNIVERSITY COLLEGE, UNIVERSITY OF LONDON.

DRAPERS' COMPANY RESEARCH MEMOIRS.

TECHNICAL SERIES, I.

ON A THEORY OF THE STRESSES IN CRANE AND COUPLING HOOKS WITH EXPERIMENTAL COMPARISON WITH EXISTING THEORY.

BY

E. S. ANDREWS, B.Sc.Eng., Lond.,
UNIVERSITY COLLEGE, LONDON.

WITH SOME ASSISTANCE FROM

KARL PEARSON, F.R.S.,
PROFESSOR OF APPLIED MATHEMATICS AND MECHANICS, UNIVERSITY COLLEGE, LONDON.

[WITH THIRTEEN DIAGRAMS.]

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On a Theory of the Stresses in Crane and Coupling Hooks with Experimental Comparison with Existing Theory.

By E. S. ANDREWS, *B.Sc.Eng., Lond., University College, London.*

With some Assistance from KARL PEARSON, *F.R.S., Professor of Applied Mathematics and Mechanics, University College, London.*

[WITH THIRTEEN DIAGRAMS.]

(1.) *Introductory.*

THE full treatment of the equations of elasticity, as applied to the strains in an elastic body of such a complex shape as a crane or coupling hook, is with our present mathematical knowledge quite beyond analytical treatment. An elementary discussion which treats the hook as a rib or even as a beam has found its way into the elementary text books,* and is possibly due originally to BRIX.† GRASHOF later than BRIX ('Theorie der Elasticität und Festigkeit,' Berlin, 1878, p. 289) has a better but still quite fallacious treatment not only of the hook but of the link problem. WINKLER, in his paper of 1858, on links of chains ("Formänderung und Festigkeit gekrümmter Körper, insbesondere Ringe," 'Der Civilingenieur,' Bd. IV., S. 232–46, 1858), corrected one great error of BRIX's investigation, and showed that the flexural rigidity of a link involves not simply the radius of gyration of the cross-section, but also its unstrained curvature. WINKLER's paper has been corrected and extended by PEARSON ('History of Elasticity,' vol. 2, Part I., pp. 422–445), but, whether applied to hooks or links, it is still theoretically fallacious and leads to results which experimentally are far from verified. In the course of the present investigation we shall show where these theories fail.

* W. C. UNWIN, 'Elements of Machine Design,' Part I., Art. 297; J. GOODMAN, 'Mechanics applied to Engineering,' pp. 383–5. The treatment as a cranked tie-bar is, of course, illegitimate, as it neglects the curvature of the belly.

† 'Verhandlungen des Vereins zur Beförderung des Gewerbfleisses in Preussen,' Jahrgang 24, Berlin, 1845, pp. 185–192. See TODHUNTER and PEARSON, 'History of Elasticity,' vol. 1, p. 675.

(2.) *General Theory.*

Our assumptions will be of the following kind :—

(*a.*) We confine our attention to the principal or horizontal section of the hook, *i.e.*, the section AB perpendicular to the line XX of loading. The line of loading meets this section produced in L, the "load-point" of the cross-section. If C be the centroid of the principal cross-section, and O the centre of curvature of the central

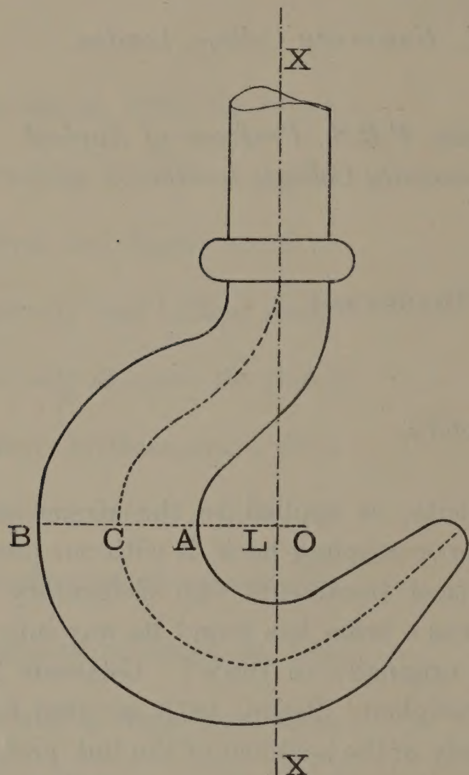


Fig. I.

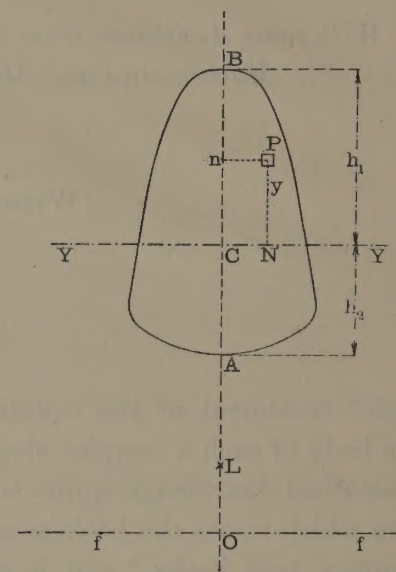


Fig. II.

line, *i.e.*, the line of centroids of successive cross-sections of the hook, we shall take $CO = \rho$ the radius of curvature. In the actual crane and coupling hooks we have measured, L and O very closely coincide. This is not necessary, but it to some extent simplifies the formulæ.

(*b.*) We shall assume that the centre of curvature of the extrados of the hook at B and of the intrados at A are sensibly at O. This is not absolutely true, especially in the case of coupling hooks, but it is quite a close approximation to the truth.

(*c.*) The curvature changes very slightly in the neighbourhood of ACB. This is quite true, for the drawings of hooks we have received show that this portion of both extrados and intrados are struck off with circles.

(*d.*) There is no shear on the section ACB, and the shear as we pass from the section ACB only slowly begins to be sensible. Hence in the *immediate* neighbour-

hood of the section ACB we may neglect any distortion of the cross-section, as far as it is due to shear, *i.e.*, we may consider the section after strain to remain plane.

We shall use symbols and letters with the subscript zero attached to them to mark the quantities and points on the unstrained hook. Without this subscript they will refer to the strained hook. Fig. II. is a picture of the cross-section ACB. The line YY, perpendicular to the trace ACB of the plane of loading on the cross-section, is the "central axis." α is any small element of area at the point P, distant y from the central line. Pn and PN are perpendiculars on BCA and YY. L is the load-point, O the centre of curvature, and $\mathrm{OC} = \rho$. $h_1 = \mathrm{CB}$ and $h_2 = \mathrm{AC}$. $\mathrm{A} =$ area of principal cross-section. Let dashed letters refer to corresponding points on an adjacent cross-section obtained by taking a plane through ff, a line through O parallel to YY, and making a small angle with the plane of AYBY. If we consider a "fibre," PP′, obtained by drawing all the circular arcs with centres on $f\mathrm{O}f$ in planes perpendicular to $f\mathrm{O}f$, which pass through the boundary of α, the tension T_y in such a "fibre" will be related to the stretch s_y in it by the relation

$$\mathrm{T}_y = \mathrm{E}s_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}).$$

Here we have adopted the fundamental assumption of DE SAINT-VENANT'S theory of flexure, or we assume the transverse tractions which must be zero at the surface of the hook are zero throughout.*

Now clearly

$$s_y = (\mathrm{PP}' - \mathrm{P}_0\mathrm{P}'_0)\,\mathrm{P}_0\mathrm{P}'_0,$$

or

$$\mathrm{PP}' = \mathrm{P}_0\mathrm{P}'_0\,(1 + s_y).$$

But

$$\mathrm{PP}' = nn' \quad \text{and by} \quad (c) \text{ and } (d)$$

$$nn'/\mathrm{CC}' = (\rho + y)/\rho.$$

Further

$$\mathrm{P}_0\mathrm{P}'_0 = n_0n'_0,$$

and

$$n_0n'_0/\mathrm{C}_0\mathrm{C}'_0 = (\rho_0 + y_0)/\rho_0.$$

Let $s_0 =$ stretch of central line at C, or

$$\mathrm{CC}' = \mathrm{C}_0\mathrm{C}'_0\,(1 + s_0).$$

Then we have the fundamental relation

$$\left(1 + \frac{y}{\rho}\right)(1 + s_0) = \left(1 + \frac{y_0}{\rho_0}\right)(1 + s_y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.}).$$

* This is probably correct to a higher degree of approximation for our principal section without shear than it is even for DE SAINT-VENANT'S cantilevers with constant shear on each cross-section.

This result with our assumptions is true for beams, girders, ribs, hooks, links, &c., perfectly generally.

If we make $\rho_0 = \infty$, and ρ very large, y/ρ will be small, and $s_0 \times \frac{y}{\rho}$ is negligible. We then have

$$s_y = s_0 + \frac{y}{\rho}.$$

This is the ordinary result for the theory of beams. But ρ_0 is not infinite for a hook; hence when BRIX and others apply such a result to the theory of the hooks, they reach results absolutely fallacious both experimentally and theoretically.

If $1/\rho_0$ be not zero, but both ρ and ρ_0 large as compared with the linear dimensions of the cross-section, we may neglect the product of the stretches s_y and s_0 into $\frac{y}{\rho}$ and $\frac{y_0}{\rho_0}$.

Hence we find

$$s_y = s_0 + \frac{y}{\rho} - \frac{y_0}{\rho_0}.$$

Now

$$\frac{y}{\rho} = \frac{y_0}{\rho} + \frac{y_0 - y}{\rho} = \frac{y_0}{\rho} + \frac{y_0}{\rho}\left(\frac{y_0 - y}{y_0}\right).$$

But $\frac{y_0 - y}{y_0}$ is of the order of the transverse squeeze and therefore may be neglected, when multiplied by $\frac{y_0}{\rho}$. Thus finally we have

$$s_y = s_0 + y_0\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right).$$

This is the legitimate formula for *flat* arches. It can never be applied to links of chains and hooks, as has been done by GRASHOF and other writers. In a hook y is of the order ρ, in fact it is quite possible for y/ρ to be unity, hence its product with the stretches cannot be neglected.

If we do not consider y/ρ as small, we have from (ii.)

$$\left(1 + \frac{y}{\rho}\right)\Big/\left(1 + \frac{y_0}{\rho_0}\right) = (1 + s_y)/(1 + s_0) = 1 - s_0 + s_y.$$

Or

$$s_y = s_0 + \frac{\frac{y}{\rho} - \frac{y_0}{\rho_0}}{1 + \frac{y_0}{\rho_0}} \quad \ldots \ldots \ldots \ldots \quad \text{(iii.)}.$$

This is the correct formula for links of chains and hooks.

WINKLER, followed by BACH* and others, introduces the denominator $1 + y_0/\rho_0$ into the flat arch formula, and puts

$$s_y = s_0 + \frac{y_0\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)}{1 + \frac{y_0}{\rho_0}}.$$

Their theory is therefore much better than that of BRIX and the elementary text-books. It is still, however, erroneous, because it replaces y by y_0 or neglects a term $\frac{y - y_0}{y_0}$, which is not multiplied by a small quantity and is of the order of s_0 and s_y, the terms naturally retained.

Our present theory retains the term $(y - y_0)/y_0$ and thus uses the complete and necessary expression (iii.) for the stretch s_y.

Let $\eta =$ POISSON's ratio, then we know that without transverse stresses the transverse squeeze $= -\eta \times s_y$. But the transverse stretch in the neighbourhood of P is $(\delta y - \delta y_0)/\delta y_0$. Thus we find

$$\delta y = \delta y_0 (1 - \eta s_y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iv.}).$$

By differentiating (iii.) we shall be able to eliminate δy and find a relation between s_y and y_0.

We have from (iii.)

$$(s_y - s_0)\left(1 + \frac{y_0}{\rho_0}\right) = \frac{y}{\rho} - \frac{y_0}{\rho_0}.$$

Hence

$$\delta s_y\left(1 + \frac{y_0}{\rho_0}\right) + (s_y - s_0)\,\frac{\delta y_0}{\rho_0} = \frac{\delta y}{\rho} - \frac{\delta y_0}{\rho_0}.$$

Using (iv.) and rearranging, we find

$$\frac{\delta s_y}{\frac{1}{\rho} - \frac{1}{\rho_0} + \frac{s_0}{\rho_0} - s_y\left(\frac{1}{\rho_0} + \frac{\eta}{\rho}\right)} = \frac{\delta y_0}{1 + \frac{y_0}{\rho_0}}.$$

In the term involving s_y we may put $\rho = \rho_0$, as the difference of $\frac{1}{\rho}$ and $\frac{1}{\rho_0}$ is very small and its product with s_y may be neglected. Thus, by integration,

$$\rho_0 \log\left(1 + \frac{y_0}{\rho_0}\right) + \frac{\rho_0}{1 + \eta} \log\left(\frac{1}{\rho} - \frac{1}{\rho_0} + \frac{s_0}{\rho_0} - \frac{(1 + \eta)\, s_y}{\rho_0}\right) = \text{const.},$$

or,

$$\frac{1}{\rho} - \frac{1}{\rho_0} + \frac{s_0}{\rho_0} - \frac{(1 + \eta)\, s_y}{\rho_0} = \frac{\text{const.}}{\left(1 + \frac{y_0}{\rho_0}\right)^{1+\eta}}.$$

* C. BACH, 'Elasticität und Festigkeit,' Berlin, 1890, see pp. 311 *et seq.*

To determine the constant, we note that $s_y = s_0$ when $y_0 = 0$, hence:

$$\text{Const.} = \frac{1}{\rho} - \frac{1}{\rho_0} - \frac{\eta s_0}{\rho_0}.$$

Thus finally

$$s_y = \frac{s_0}{1+\eta} + \frac{\rho_0}{1+\eta}\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) - \frac{\frac{\rho_0}{1+\eta}\left(\frac{1}{\rho} - \frac{1}{\rho_0} - \frac{\eta s_0}{\rho_0}\right)}{\left(1 + \frac{y_0}{\rho_0}\right)^{1+\eta}},$$

or,

$$s_y = s_0 + \left\{\frac{\rho_0}{1+\eta}\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) - \frac{\eta s_0}{1+\eta}\right\}\left(1 - \frac{1}{\left(1 - \frac{y_0}{\rho_0}\right)^{1+\eta}}\right) \quad . \quad . \quad . \quad \text{(v.)}.$$

This gives us for the tensile stress T_y at distance y from the central line

$$T_y = Es_0 + E\left\{\frac{\rho_0}{1+\eta}\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) - \frac{\eta s_0}{1+\eta}\right\}\left(1 - \frac{1}{\left(1 + \frac{y_0}{\rho_0}\right)^{1+\eta}}\right) \quad . \quad . \quad \text{(vi.)}.$$

But we have at once* $Q = S(T_y\alpha_0)$ and $Qc = -S(T_y\alpha_0 \times y_0)$ by considering the equilibrium of the part of the hook below the principal cross-section, Q being here the load on the hook, and Qc the bending moment at this section and S a summation for all elements α of A. Writing for brevity

$$\beta = \frac{\rho_0}{1+\eta}\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) - \frac{\eta s_0}{1+\eta},$$

we find

$$Q = EAs_0 + E\beta A(1 - \gamma_1)$$

$$-Qc = +E\beta A\rho_0\gamma_2,$$

where

$$A\gamma_1 = S\left\{\frac{\alpha_0}{\left(1 + \frac{y_0}{\rho_0}\right)^{1+\eta}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii.)},$$

$$A\gamma_2 = -S\left\{\frac{\alpha_0\frac{y_0}{\rho_0}}{\left(1 + \frac{y_0}{\rho_0}\right)^{1+\eta}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(viii.)}.$$

Thus we have

$$\beta = -\frac{Qc}{EA\rho_0\gamma_2},$$

$$s_0 = \frac{Q}{EA} + \frac{Qc}{EA\rho_0}\,\frac{1 - \gamma_1}{\gamma_2}.$$

* We may here use α_0 and z indifferently, as they are multiplied by the small quantities s_0 and $1/\rho - 1/\rho_0$.

Substituting for β, we determine

$$\frac{1}{\rho} - \frac{1}{\rho_0} = \frac{Q}{EA}\left\{-\frac{c(1+\eta\gamma_1)}{\gamma_2\rho_0^2} + \frac{\eta}{\rho_0}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ix.)},$$

$$s_0 = \frac{Q}{EA}\left\{1 + \frac{c}{\rho_0}\,\frac{1-\gamma_1}{\gamma_2}\right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(x.)}.$$

These give, when γ_1 and γ_2 have been found, the change in curvature and the stretch in the central line at the principal cross-section in terms of the pull on the hook.

Substituting for s_0 and β in (vi.), we have

$$T_y = \frac{Q}{A}\left\{1 + \frac{c}{\rho_0\gamma_2}\left(\frac{1}{\left(1+\frac{y_0}{\rho_0}\right)^{1+\eta}} - \gamma_1\right)\right\} . \quad . \quad . \quad . \quad . \quad . \quad \text{(xi.)}.$$

It is clear that the stress is a function only of the distance from the central line, and that it will reach maximum values—without regard to sign—when y_0 takes its maximum positive and negative values. Let C_{h_1} be the maximum compressive stress, *i.e.*, the stress at the extrados, and T_{h_2} the maximum tensile stress, *i.e.*, the stress at the intrados, then we have, since γ_1 is usually greater than unity, and $1 + h_2/\rho_0$ and $1 - h_1/\rho_0$ respectively greater and less

$$C_{h_1} = \frac{Q}{A}\left\{\frac{c}{\rho_0\gamma_2}\left(\gamma_1 - \frac{1}{\left(1+\frac{h_1}{\rho_0}\right)^{1+\eta}}\right) - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(xii.)},$$

$$T_{h_2} = \frac{Q}{A}\left\{\frac{c}{\rho_0\gamma_2}\left(\frac{1}{\left(1-\frac{h_2}{\rho_0}\right)^{1+\eta}} - \gamma_1\right) + 1\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(xiii.)}.$$

There will be a true neutral axis, the distance $\bar{y}_0$ of which from the central line is given by

$$\bar{y}_0 = \rho_0\left\{\frac{1}{\left(\gamma_1 - \frac{\rho_0}{c}\gamma_2\right)^{\frac{1}{1+\eta}}} - 1\right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)}.$$

The stresses, however, will not vary simply as the distance from this neutral line.

Results (xii.)–(xiv.) reduce to the customary formulæ for arches or beams, if ρ_0 be considerable as compared with y_0, namely,

$$C_{h_1} = \frac{Q}{A}\left(\frac{h_1 c}{k^2} - 1\right), \quad T_{h_2} = \frac{Q}{A}\left\{\frac{ch_2}{k_2} + 1\right\} \text{ and } \bar{y}_0 = k^2/c \;. \quad . \quad . \quad \text{(xv.)}.$$

This follows when we notice that for y_0/ρ_0 small, (vii.) and (viii.) give us

$$\gamma_1 = 1, \quad \gamma_2 = (1+\eta)\,k^2/\rho_0^2,$$

where Ak^2 is the second moment of the area of the principal cross-section about the central line.

(xv.) are the formulæ improperly given in the text-books for the strength of hooks. They are quite independent of ρ_0, and therefore, according to them, the strength of a hook is absolutely independent of the curvature of its belly. For example, a curved and a straight-bellied hook ought to be equally strong. This is a point which certainly admits of experimental investigation. Formulæ (xv.) give very irregular results for both crane and coupling hooks,* even under moderate working loads. We must investigate whether (xii.) and (xiii.) lead to more uniform values.

(3.) *On the determination of γ_1 and γ_2.*

First Method.—Clearly the discussion of the stresses will be perfectly straightforward—supposing Poisson's ratio known—if we can find γ_1 and γ_2 for any cross-section.

Now,

$$A\gamma_2 = -S\left(\frac{(y_0/\rho_0)\,\alpha_0}{(1+y_0/\rho_0)^{1+\eta}}\right) = -S\left(\frac{\alpha_0}{(1+y_0/\rho_0)^{\eta}}\right) + A\gamma_1.$$

Let

$$A\gamma_3 = S\left(\frac{\alpha_0}{(1+y_0/\rho_0)^{\eta}}\right),$$

then

$$\gamma_2 = \gamma_1 - \gamma_3,$$

and the solution depends upon the determination of

$$A\gamma_1 = S\left\{\left(\frac{\rho_0}{\rho_0+y_0}\right)^{1+\eta}\alpha_0\right\} \text{ and } A\gamma_3 = S\left\{\left(\frac{\rho_0}{\rho_0+y_0}\right)^{\eta}\alpha_0\right\} \quad . \quad . \quad \text{(xvi.)}.$$

Both of these expressions fall under the form

$$A\gamma' = S\left\{\left(\frac{\rho_0}{\rho_0+y_0}\right)^{n}\alpha_0\right\}.$$

Suppose the curve $z = \epsilon \times Y^n$ to be drawn, say, cut out as a template for a selected value or values of n. Then we can get a very simple construction to determine γ'.

Let the Fig. III. represent a section. Place the template with its Y base against the axis of symmetry OACB and draw the curve $z = \epsilon \times Y^n$ from its edge, *i.e.*, OJ_0JH, with its origin at the centre of curvature O.

Let a strip PP′ of the area of the cross-section parallel to YY, the central line, meet this curve in J, and the central line meet it in J_0; then we have

$$CJ_0 = \epsilon \times \rho_0^{\,n}, \quad nJ = \epsilon \times (\rho_0 + y_0)^n,$$

and

$$A\gamma' = S\left\{\left(\frac{CJ_0}{nJ}\right) \times \text{area of strip PP}'\right\}.$$

* See, for example, some interesting experimental work on crane hooks by Professor Goodman, who applies the old theory: 'Engineering, Vol. LXXII., pp. 537 *et seq.*

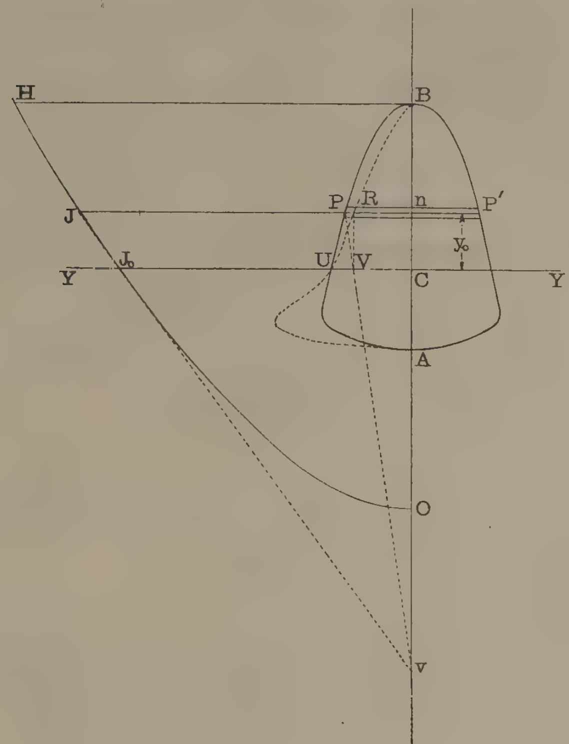


Fig. III.

Let the chord JJ_0 meet BCAO in v; let vP meet YY, the central line, in V and VR be the perpendicular on PP′, then $Rn = VC = Pn \times \frac{CJ_0}{nJ}$.

Thus:

$$A\gamma' = 2 \times S \text{ (strips of area like } Rn).$$

Let a series of points like R give the dotted curve BRUA, then clearly $A\gamma'$ is equal to twice the area of the curve BRUA. We call this a hook-rigidity curve. We shall speak of γ_1 and γ_3 as being given by twice the areas of the hook-rigidity curves divided by the area of the section. So soon as the hook-rigidity curves for any principal section have been constructed, their areas can be found by a planimeter, and the stresses at this principal section will then be known.

The two curves that we require to draw are

$$z = \epsilon \times Y^{1+\eta} \quad \text{and} \quad z = \epsilon \times Y^{\eta}.$$

We ought properly to make an experiment to find η for the material of our hook, which will generally be wrought iron or steel. Such an experiment will not usually be possible, and we shall not be far wrong for these materials if we take $\eta = \frac{1}{4}$. In this case our two curves will be

$$z = \epsilon \times Y^{5/4} \quad \text{and} \quad z = \epsilon \times Y^{1/4}.$$

Second Method.—The following special method is suggested for finding the two hook-rigidity curves in this case :—

Describe any parabola with O as vertex and OACB as axis: see Fig. IV. Let it be OV. Describe any second parabola with O as vertex and OY′ parallel to YCY as axis. Let it be OW, and be such that TW = a suitable length, say 8 inches. Let $BV = a$, $VT = d$, $TW = b$. Let any line of the hook section PnP', parallel to YY, meet the first parabola in v and vwt be parallel to BCAO, then, if $tw = z$,

$$z^2/b^2 = \mathrm{O}t/a, \text{ or } z^4/b^4 = \mathrm{O}t^2/a^2 = nv^2/a^2.$$

But

$$nv^2/a^2 = \mathrm{O}n/\mathrm{OB} = \mathrm{O}n/d.$$

Therefore

$$z^4/b^4 = \mathrm{O}n/d.$$

Thus, $z = b\left(\frac{\mathrm{O}n}{d}\right)^{1/4}$ or if tw or z be plotted to $\mathrm{O}n$ we have the first required curve

$$z = \epsilon \times (\mathrm{Y})^{1/4}.$$

Let YY meet this curve in J_0 and $P'nP$ in J. Let the vertical JI through J meet YY in I. Let IP meet AB in i and $J_0 i$ meet $\mathrm{P}n$ in R. Then

$$\mathrm{R}n/\mathrm{P}n = \mathrm{J_0C/IC} = \mathrm{J_0C}/\mathrm{J}n = \frac{\mathrm{OC}^{1/4}}{\mathrm{O}n^{1/4}}.$$

Thus

$$\mathrm{R}n = \mathrm{P}n \times \left(\frac{\rho_0}{\rho_0 + y_0}\right)^{1/4},$$

and R gives the hook-rigidity curve $\mathrm{BRUR_1A}$.

Join RO and let it meet VC in m, draw mS perpendicular to PP′; then

$$\mathrm{S}n/\mathrm{R}n = m\mathrm{C}/\mathrm{R}n = \frac{\rho_0}{\rho_0 + y_0}.$$

Thus

$$\mathrm{S}n = \frac{\rho_0}{\rho_0 + y_0}\,\mathrm{R}n = \mathrm{P}n \times \left(\frac{\rho_0}{\rho_0 + y_0}\right)^{5/4},$$

or S gives the second hook-rigidity curve $\mathrm{BSUS_1A}$.

$$\gamma_1 = \frac{\text{area } \mathrm{BSUS_1A}}{\text{area } \mathrm{BPUP_1A}},$$

$$\gamma_3 = \frac{\text{area } \mathrm{BRUR_1A}}{\text{area } \mathrm{BPUP_1A}},$$

$$\gamma_2 = \gamma_1 - \gamma_3.$$

This method gives better results than that described above, as we find points like v of Fig. III. may be at considerable distances. The curve $\mathrm{OJ_0J}$ can be drawn once for all and a template made.

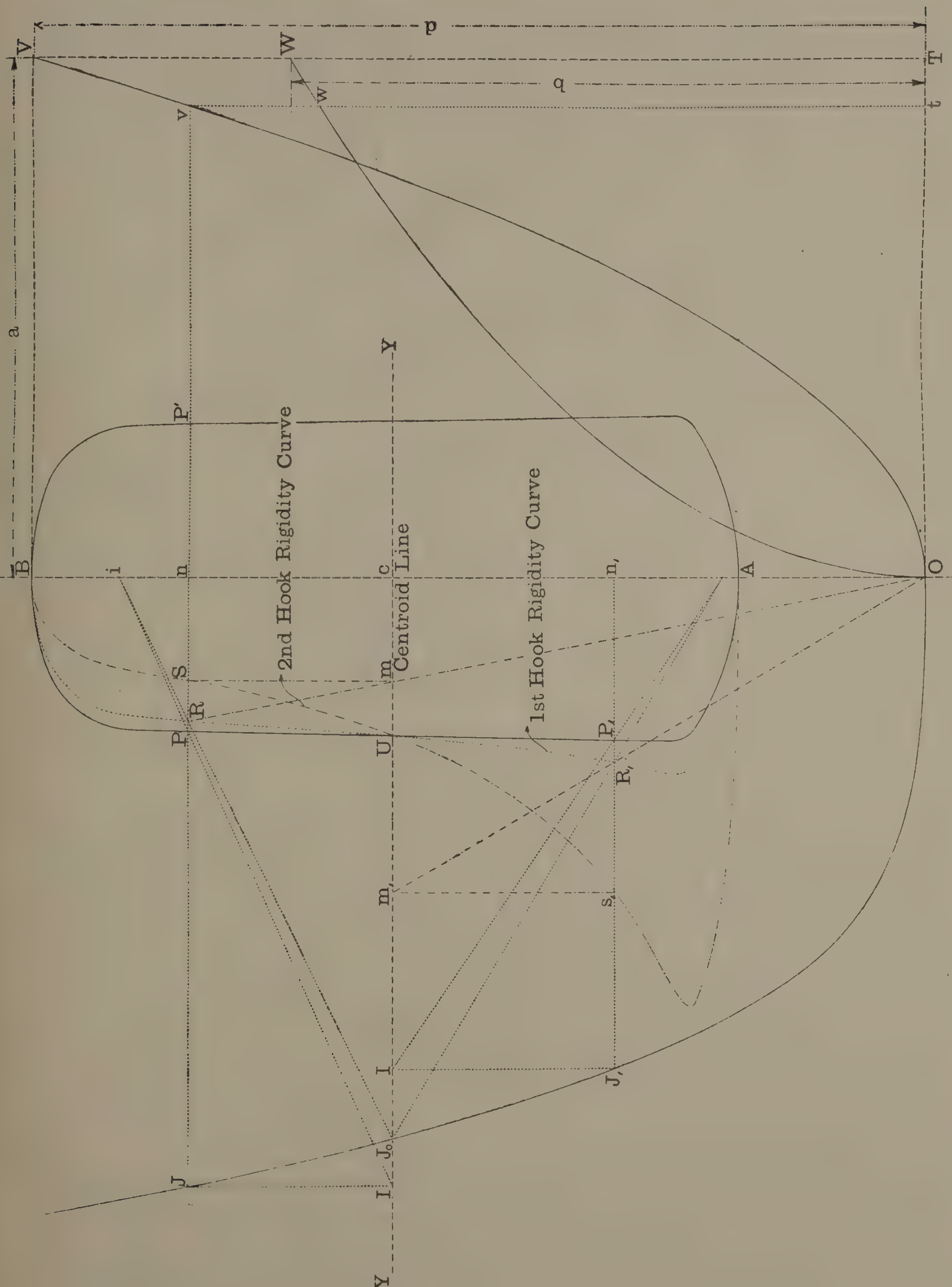


Fig. IV. Much reduced from actual graphical treatment of a railway wagon coupling hook.

Special Case.—It will be seen in the sequel that for comparative experimental purposes it has been found advisable to deal with a type of hook that would be of small service for practical purposes, namely, a hook with rectangular principal cross-section. In this case γ_1 and γ_3 can be found by analysis. Suppose the section to be of thickness 2τ in the planes of loading and of breadth f.

Then

$$\mathrm{A}\gamma_1 = \int_{-\tau}^{+\tau} \left(\frac{\rho_0}{\rho_0 + y_0}\right)^{1+\eta} f\,dy_0,$$

$$= \frac{f}{\eta}\left[\rho_0^{\,1+\eta} \times \left\{-\frac{1}{(\rho_0 + y_0)^\eta}\right\}\right]_{-\tau}^{+\tau},$$

$$= \frac{f\rho_0}{\eta}\left\{\left(\frac{\rho_0}{\rho_0 - \tau}\right)^\eta - \left(\frac{\rho_0}{\rho_0 + \tau}\right)^\eta\right\}.$$

Hence

$$\gamma_1 = \frac{\rho_0}{\eta \,.\, 2b}\left\{\left(\frac{\rho_0}{\rho_0 - \tau}\right)^\eta - \left(\frac{\rho_0}{\rho_0 + \tau}\right)^\eta\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xvii.)}.$$

Similarly

$$\mathrm{A}\gamma_3 = \int_{-\tau}^{+\tau} \left(\frac{\rho_0}{\rho_0 + y_0}\right)^\eta f\,dy_0,$$

and

$$\gamma_3 = \frac{\rho_0}{(1-\eta) \,.\, 2b}\left\{\left(\frac{\rho_0 + \tau}{\rho_0}\right)^{1-\eta} - \left(\frac{\rho_0 - \tau}{\rho_0}\right)^{1-\eta}\right\}. \quad . \quad . \quad . \quad . \qquad \text{(xviii.)},$$

while

$$\gamma_2 = \gamma_1 - \gamma_3, \text{ as before} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xix.)}.$$

(4.) *On the Influence of* Poisson's *Ratio on the Values of the Stresses.*

It will be seen from the above results that the actual values of the stresses depend upon the magnitude of Poisson's ratio. According to the French school of elasticians the value of η for all isotropic materials should be ·25. Now the difficulties of determining η by direct experiment are well known, and even when we proceed indirectly by tensile and torsional moduli some doubts may be legitimately felt as to the closeness of the degree of accuracy with which η is found. Values of η obtained in this way for wrought iron or mild steel give in general something about ·28. Thus the probable range of η is almost certain to be included between ·25 and ·30.

Now, in order to try the effect of such variation in η on the values of the stresses calculated by the above theory, very careful calculations of those stresses were made on the principal section of an actual railway crane hook, first for $\eta = {}$·25 and then for $\eta = {}$·30. The values of γ_1 and γ_3 were determined by quadrature from the integrals (xvi.), thus avoiding any error of draughtsmanship in using the methods of Section (3). The difference in the maximum stresses in the two cases came to 3 per cent. It would thus appear that if η be actual ·28 and not ·25 for our material, the stresses are not likely to be more than 1 to 2 per cent, in error owing to

assuming η to be ·25. Remembering the difficulties of really good determinations of η, and the *practically* insignificant deviations of 1 to 2 per cent. arising from admissible changes in its value, we have thought it far better to suppose $\eta = \cdot 25$ throughout, a value which at any rate has a good deal of theoretical justification for steel of fairly homogeneous character.

(5.) *Experimental Work.*

The importance of testing the present theory lies principally in the following divergences between the old theory and the new :

(*a.*) The old theory in the form given by BRIX and UNWIN makes the strength of the hook depend in no way upon the curvature of its belly. If this were true, the strength of the hook would be independent of the shape of its central line in the neighbourhood of the principal cross-section. There would be no means therefore of strengthening a hook by a good shape of belly.

(*b.*) The old theory makes very insignificant differences between the maximum compressive and tensile stresses at the principal section. See, for example, GOODMAN'S results for crane hooks.* We could not hope therefore—if we suppose the safe stresses for wrought iron or mild steel to be fairly equal—to better sensibly the form of the existing cross-sections.

The new theory makes the maximum tensile stress very sensibly larger than the maximum compressive stress in any crane or coupling hooks which have so far been worked out.

Now there is no doubt that the dimensions of both crane and coupling hooks have to be largely determined by the effect of wear in use.† But this shaping for wear does not largely affect the shape of the hook in the neighbourhood of the principal cross-section. Hence, if we look upon the principal section as the weakest part of the hook, it would follow that if the present theory be correct, it is possible to design a hook with a better cross-section than those at present in use.‡

* *Loc. cit.*, p. 538.

† This was exemplified in the railway wagon coupling hooks which have been tested, some being new and some worn hooks.

‡ If we wish to design such a cross-section for given area A and given curvature of belly $1/\rho_0$, we must make the C_{h_1} of (xii.) equal to the T_{h_2} of (xiii.). This gives us

$$\gamma_1 - \gamma_2\rho_0/c = \tfrac{1}{2}\{(1 - h_2/\rho_0)^{-(1+\eta)} + (1 + h_1/\rho_0)^{-(1+\eta)}\}$$

as the desirable relation between the h's and the γ's. Of course this is somewhat complex and could only be approximated to by trial and error.

If we take the load at the centre of curvature, *i.e.*, $\rho_0 = c$, the left-hand side becomes γ_3, or :

$$\frac{1}{A}\,S\,\{\alpha_0(1 + y_0/\rho_0)^{-\eta}\} = \tfrac{1}{2}\{(1 - h_2/\rho_0)^{-(1+\eta)} + (1 + h_1/\rho_0)^{-(1+\eta)}\}.$$

As to (*a.*) above, it has practically been recognised as erroneous by those who, like BACH, follow the incomplete theory of WINKLER. To test it roughly, six vulcanite "hooks" were made with identical rectangular cross-sections. Three of the hooks were split rings or of uniform curvature; the other three were split links, *i.e.*, they had the same curvature as the rings at the ends, but were straight at the belly. We should on our theory expect the curved belly to be weaker than the straight belly, and further, the hook in the latter case to give just where the curved part began. Both rings and hooks were loaded by gradually running water into a tank suspended from them until they broke. The rings gave at the principal section and the links at the junction of curved and straight pieces without exception. This experiment is for several reasons inconclusive, but at any rate it suggests that we cannot afford, as in the old theory, to neglect the curvature of the belly.

The principal defect in such an experiment is the extension of the pure elastic theory of stress, right up to rupture. The problem arising in our present case is essentially that which occurs when so many problems in elasticity are dealt with experimentally. The caution required in observation, and the delicacy of the recording apparatus needed in determining the limit of proportionality of stress and strain, are too often overlooked by experimenters, who content themselves with noticing the stress at which the far more easily observed "yield" occurs. It appears to us that it is the difference in the yield-points of tensile and flexural specimens, not the difference in their true elastic limits, which forms the old "beam paradox" which has much troubled some engineers. If the deflections in a beam are very carefully measured, we believe the elastic limit will be found to be in accordance with that obtained in a pure tensile experiment on the same material, but the yield-point will not occur until the calculated stress is about 30 per cent. higher. Nor is the reason for this hard to seek. The yield-point is the point at which there is a sudden and rapid increase of strain without corresponding increase of stress.* When stressed up to the yield-point, a tensile specimen is equally stressed at *all* points, and may begin yielding everywhere. On the other hand, a flexural specimen has its maximum stress at one point of one section only, and a yielding at this point is not

Expanding, after some reductions, we find, if $A\mu^3 = S(y_0^3\alpha_0)$, and $A\nu^4 = S(y_0^4\alpha_0)$,

$$\eta(k/\rho_0)^2 - \tfrac{1}{3}\eta(2+\eta)(\mu/\rho_0)^3 + \tfrac{1}{12}\eta(2+\eta)(3+\eta)(\nu/\rho_0)^4 + \&c.$$
$$= (h_2 - h_1)/\rho_0 + \tfrac{1}{2}(2+\eta)(h_1^2 + h_2^2)/\rho_0^2 + \tfrac{1}{6}(2+\eta)(3+\eta)(h_2^3 - h_1^3)/\rho_0^3$$
$$+ \tfrac{1}{24}(2+\eta)(3+\eta)(4+\eta)(h_2^4 + h_1^4)/\rho_0^4 + \&c.$$

But μ/ρ_0 will be small and approximately we shall have

$$\tfrac{1}{4}k^2/\rho_0^2 = (h_2 - h_1)/\rho_0 + \tfrac{9}{8}(h_1^2 + h_2^2)/\rho_0^2, \quad \text{or,} \quad k^2 = 4\rho_0(h_2 - h_1) + 4{\cdot}5(h_1^2 + h_2^2).$$

For example, an isosceles triangle, if the radius of curvature of the belly were 1·75 its height, would roughly approximate to a section form of maximum strength.

* Long known and termed "limit of fatigue," of "stability," or "breakdown-point." It was termed "yield-point" in 1886, PEARSON, 'History of Elasticity,' vol. I., p. 887.

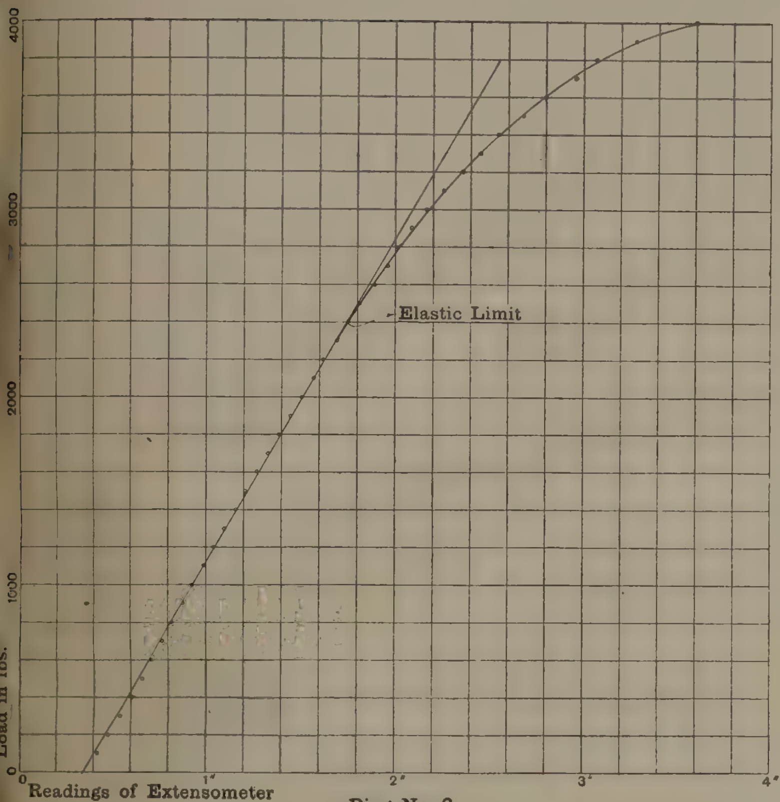
marked by the same general give or breakdown of the specimen as a whole, which occurs in the case of a bar under tension. It is not until a much larger region has reached its yield-point that a general drop in the deflection, without corresponding increase of load, becomes manifest. While for some practical purposes it may be sufficient, and for other rough and ready purposes it may be economically necessary to determine the yield-point only, and take this as an approximate measure of the maximum upper limit to elasticity, it is quite certain that the yield-point is of little or no value when one theory is to be tested against a second. All current theories are based upon the proportionality of stress to strain, and the limit of elasticity is, in the materials of the present investigation, not very far removed from this limit to the generalised HOOKE's law.* Our experience in the present case seems to show that the limit to HOOKE's law is reached at the same stress in both tensile and flexural experiments, and we have termed this in the present paper the "elastic limit." It is a limit which is quite easily shown on the stress-strain, or rather load-deformation diagrams, provided these are very carefully plotted. It is, however, quite hopeless to try and measure the increasing opening of a hook without any extensometer, as some experimenters on crane hooks have done. For in such a way we cannot hope, as experience has shown us, to get the true form of the load-deformation curve; we only succeed in getting the yield-point, and thus all our calculated stresses will be too high. Indeed, we ought not to apply theories based on the proportionality of stress and strain to yield-point stresses at all.

Hence the first series of experiments conducted at University College on a fairly large series of crane and railway coupling hooks—for which we have heartily to thank various firms of hook makers and railway companies—while providing interesting comparative material,† were ultimately seen to be of little service from the standpoint of criticising antagonistic theories. They failed to give with sufficient accuracy the true position of the elastic limit upon which everything turned. Accordingly it was, after some further experimenting, seen to be best to prepare special hooks of as homogeneous material as was procurable, and of a perfectly definite measurable shape. If the old theory fail for hooks so constructed which can be closely tested, and the new theory give concordant results, we may be pretty confident as to which is the proper theory to apply in the case of crane and coupling hooks, the structure of which is more complex, and where it is possibly harder, except in a pure tensile test, to obtain in ordinary practice the value of the elastic limit.

Accordingly a series of rings of varying cross-section and diameter were cut by the author from a piece of steel ship plate, care being taken that the direction of rolling of the plate was marked on each specimen. Two tension specimens were cut from

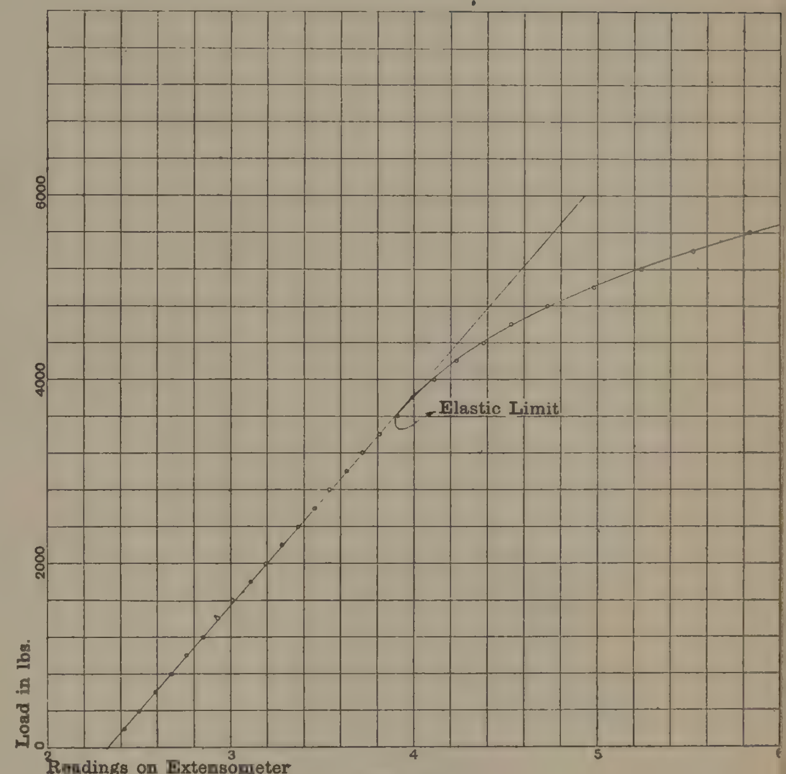
* On this point see 'History of Elasticity,' vol. I., pp. 888 *et. seq.*

† They were undertaken by Mr. H. PAYNE, then assistant in the Department, now Professor of Engineering in the South African College, Cape Town. While of great value in guiding the final experimental direction of this work, they did not provide a criterion of requisite stringency.



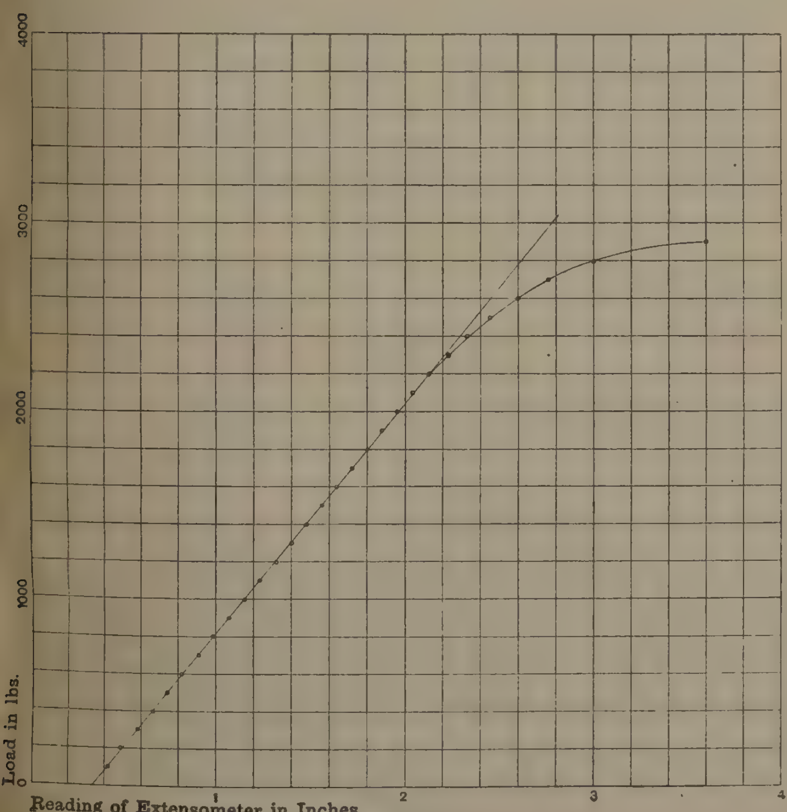
Ring No. 2

Fig. V.



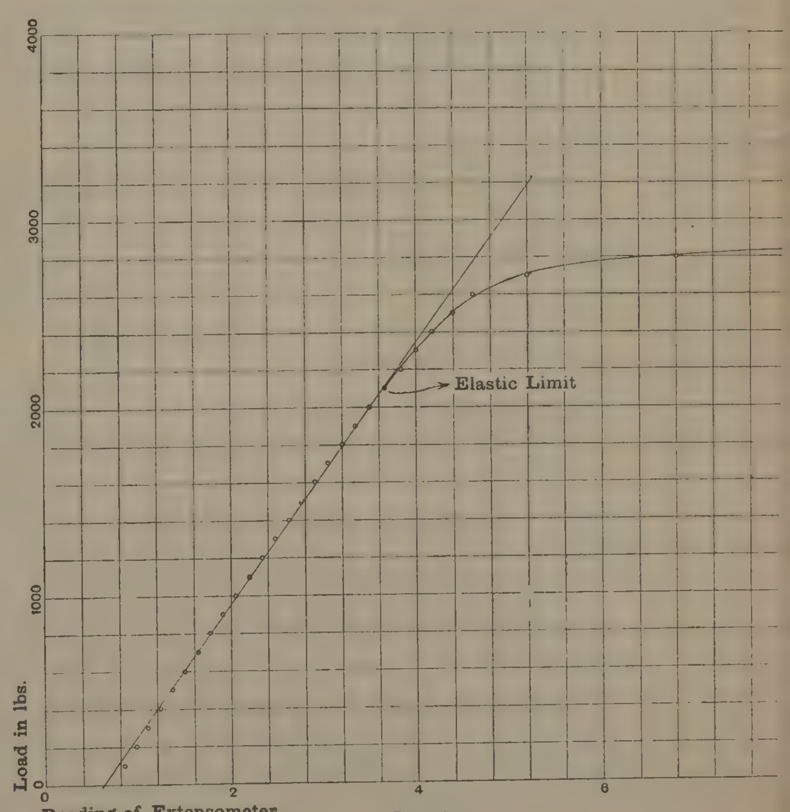
Ring No. 3

Fig. VI.



Ring No. 4

Fig. VII.



Test No. 6

Fig. VIII.

the same plate, also in the direction of rolling, in order to compare the results of the tests of the rings which were split and tested as hooks, with these specimens in direct tension.

The openings of the split ring were measured by an extensometer with a magnification of 100 : 1, and thus an opening of ·00001 inch could be detected. The load was applied in a line through the centre of curvature of the ring, its exact position being fixed by a knife edge fitting in a triangular groove. In this way a test was made and the openings plotted against the load. Such load-deformation diagrams are given in Figs. V.–VIII. for the cases of hooks Nos. 2, 3, 4 and 6 with rectangular principal section and in Figs. IX. and X. for hook No. 5, the section being T, in both tension and compression. In most kinds of wrought iron and mild steel the yield-point, in bending, is about 25 per cent. higher than the elastic limit, while in tension there is not much difference between the two. Hence, for our present purpose, when we need the limit to proportionality of stress and strain, we cannot as in tensile tests run out the jockey weight until the lever suddenly drops.* We must plot the load-deformation diagram and discover where it ceases to be linear. On referring to the diagrams, it will be seen that the load-deformation curve ceases to be linear gradually, and the exact point where it leaves the straight line is a matter for appreciation which may be more or less easy. In all cases, however, the diagrams sufficed to determine the limit of proportionality with accuracy close enough for the present investigation.

Test No. 1.—Two bars were cut, as indicated above, in the direction of rolling of the steel, and these tensile specimens were of approximately $\frac{3}{4}$ inch diameter and 10 inches between the gauge lengths. The mean stress at the elastic limit was 30,000 lbs. per sq. inch for the two bars, there being very little difference between the two results (see Fig. XI.).

The results of the tests on five hooks, Tests 2 to 6, are given in the table below.

On the old theory, if W (= Q of previous notation) be the load on the hook,

$$\mathrm{T} = \frac{\mathrm{W}}{\mathrm{A}}\left(1 + \frac{\rho h_2}{k^2}\right) = \alpha \mathrm{W},$$

gives the maximum tension at the intrados.

On the new theory by equation (xiii), if $\rho_0 = c$:

$$\mathrm{T} = \frac{\mathrm{W}}{\mathrm{A}}\left\{\frac{1}{\gamma_2}\left[\left(\frac{\rho_0}{\rho_0 - h_2}\right)^{1+\eta} - \gamma_1\right] + 1\right\} = \beta \mathrm{W}.$$

Thus α and β are the quantities by which the load must be multiplied to get the tensile stress at the intrados. They are given in Columns 9 and 10 of the table below.

* As the yielding only spreads gradually over the material, as indicated above, the "drop" is not nearly of such a marked character in bending as in tensile tests.

While the hooks in Tests 2, 3, 4 and 5 were split rings of definite radius, the hook in Test 6 had a sensibly straight belly or zero curvature at the principal cross-section. In this case the two theories give identical results.

It will be seen, on examining the table, that in all cases the stresses calculated on the new theory were nearer than those of the old to the result reached by a pure tensile test. Their fluctuation among themselves is also far less than the stresses found from the old theory. The old theory gives values ranging from 18,700 to 30,500, the new theory values from 28,500 to 30,500. Thus the fluctuation is nearly 40 per cent. in the first case as against something less than 7 per cent. in the second from the tensile elastic limit. There is no doubt about the new theory giving experimentally far better results. A like exaggerated fluctuation is found when the old theory has been applied to series of hooks of the same material, but of different sections and curvatures.

TABLE of Results of Hook-Tests.

Test number.	Section of hook.	Area of section in sq. inch.	γ_1.	γ_2.	ρ_0.	k in inches.	h_2 in inches.	α	β.	Stress at elastic limit.	
										Old theory.	New theory.
2	Rectangular	·936	1·060	·054	1·541	·302	·523	10·51	12·18	25,200	29,200
3	"	1·320	1·071	·063	1·983	·421	·730	6·92	8·03	26,300	30,500
4	"	1·360	1·019	·017	3·680	·424	·734	11·85	12·97	26,100	28,500
5	T-section	·882	1·059	·056	1·850	·424	·600	8·13	12·73	18,700	29,300
6	Rectangular	·770	1·000	0	∞	·806	·441	14·31	14·31	30,500	30,500

In Nos. 2–4 the values of γ_1 and γ_2 were calculated directly from the formulæ given above as (xvii.) to (xix.). In No. 5 a graphical construction similar to that shown in Fig. IV. was used. The dimensions of the T were, web ·94″ × ·44″, and flange ·52″ × 90″, the flange being in tension.

(6.) *Test of a Railway Wagon Coupling Hook.*

We have already stated that the bulk of the actual coupling and crane hooks collected at University College had been tested without the use of a sufficiently fine extensometer to successfully distinguish the true limit to the proportionality of stress and strain. One railway wagon hook, however, remained untested, and a complete test was made on this, subject to the necessary conditions. The load was applied through the centre of curvature of the centre line at the principal cross-section, and was located by knife edges. The section and the hook rigidity curves were dealt with graphically, Fig. IV. being the reduced drawing.

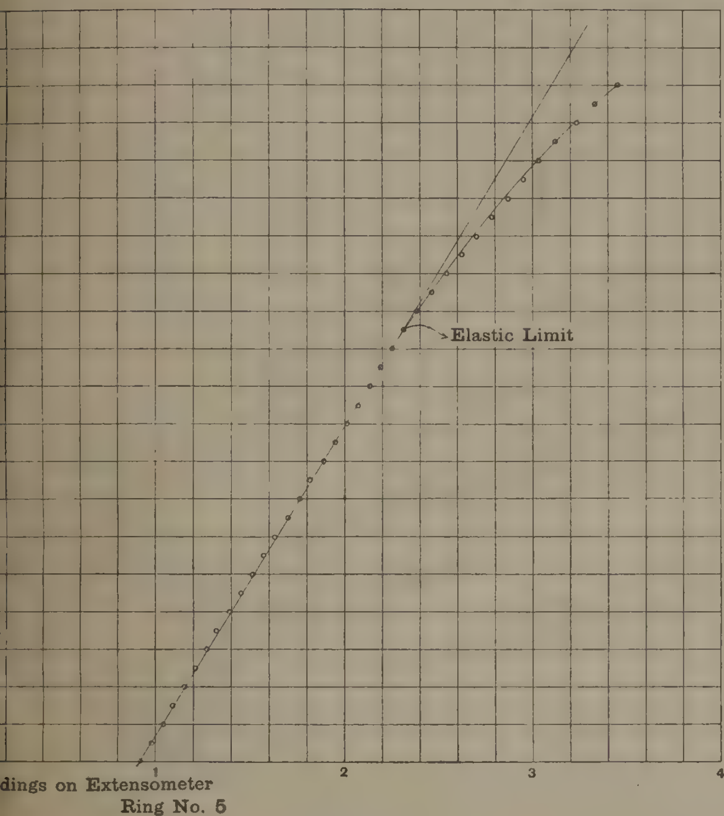


Fig. IX.

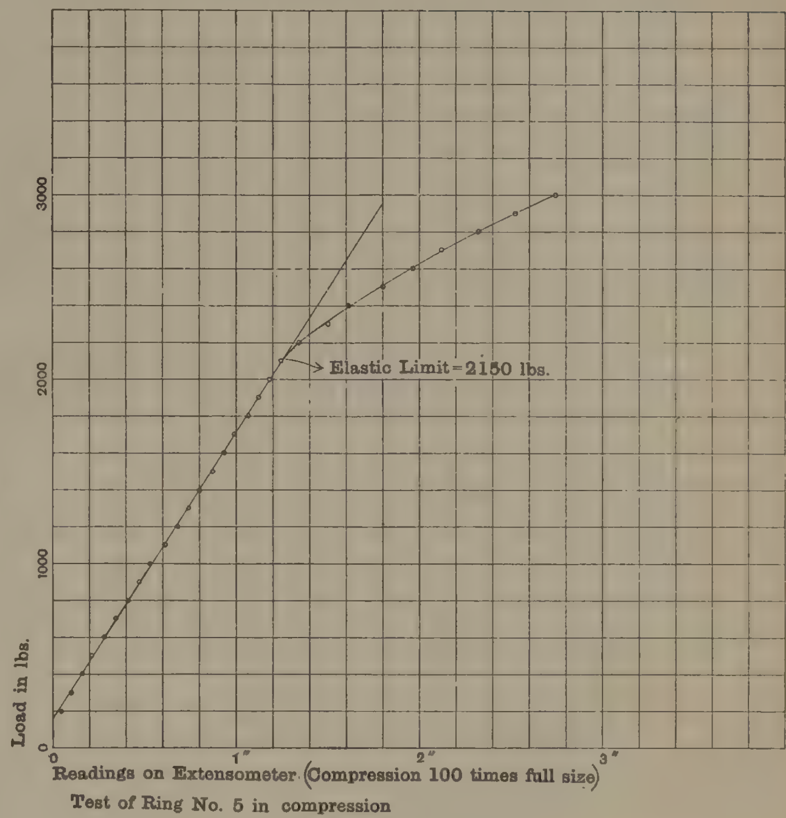


Fig. X.

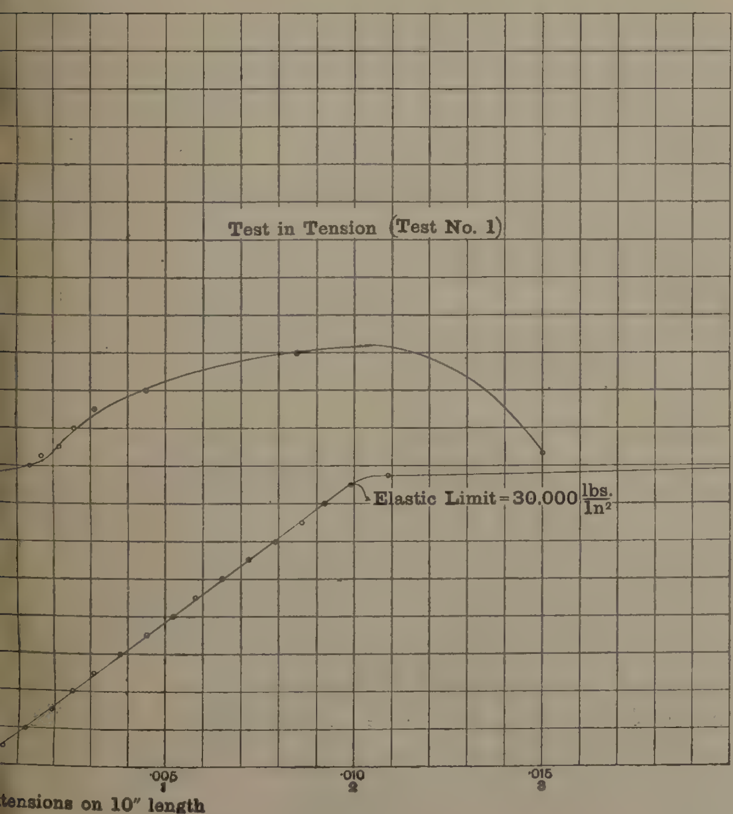


Fig. XI.

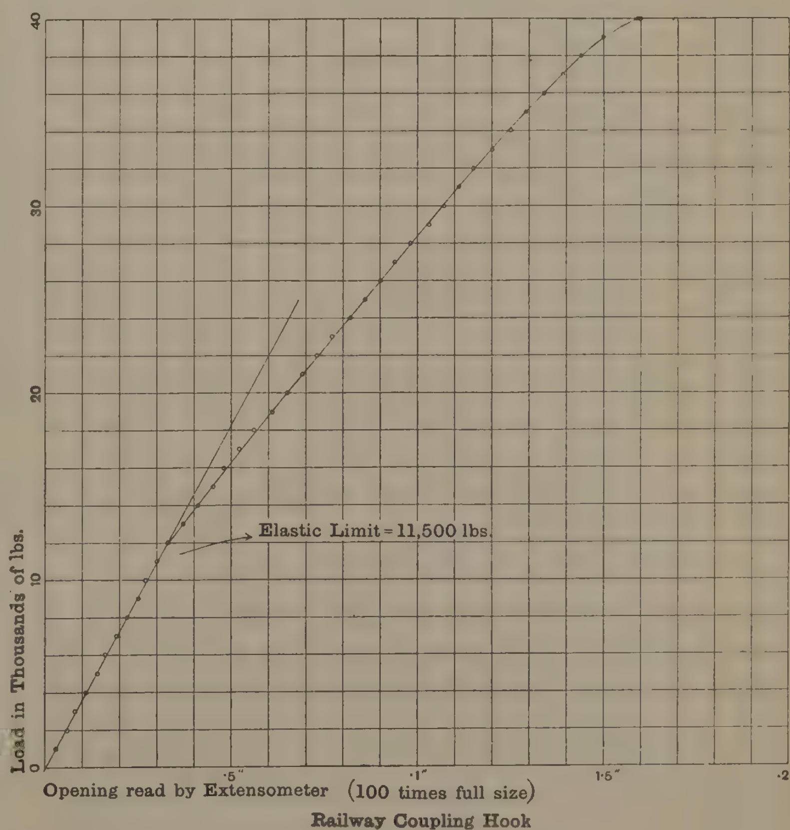


Fig. XII.

TEST of Railway Wagon Coupling Hook.

Test number.	Section.	Area of section.	γ_1.	γ_2.	ρ_0.	k.	h_1.	h_2.	α.	β.	Stress at elastic limit.	
											Old theory.	New theory.
		sq. inches.			inches.	inches.	inches.	inches.				
7	As in Fig. IV.	6·00	1·244	·221	2·84	1·01	1·90	1·84	1·04	2·01	12,000	23,000

The load at the elastic limit was 11,500 lbs. The stress-strain diagram is given in Fig. XII.

A tensile specimen cut from the unstrained part of the hook had an elastic limit of 21,000 lbs. per square inch. Thus we see that while the old theory was incorrect by 75 per cent. of its value, the new was in error by less than 9 per cent.

We have thus ample evidence that the new theory as applied to the hooks of commerce is far superior to the old. Apart from this experimental verification, however, the theoretical consideration that the radius of curvature of a hook is not indefinitely larger than the linear dimensions of its cross-section seems to show that the old beam theory, which neglects the curvature of the belly, ought never to be applied to stresses in hooks.

(7.) The whole of the present investigation depends on our theory not being extended beyond the limit of proportionality of stress and strain. It would be convenient to call this HOOKE's limit.* Usually it is close to the elastic limit, and in most structures, reduced to a state of ease, is or ought to be the limit to safe loading. This limit appears to be really the same whether it be determined for the same material by a pure tensile test, by flexure experiments, or from hooks. Thus the mean of two tensile tests gave 30,000, and of five hook tests 29,600. Further, two flexure tests were made on short bars, both 16 inches long, of the same material. The results are plotted in Fig. XIII., and show at once that the HOOKE's limit, *i.e.*, 29,000 and 30,000 for the two cases, is practically the same as for the tensile and hook tests.

Finally, to complete the series of tests, one of the rings, No. 5, was tested for the limit by compressing it between two plates, so as to *close* the opening; the contractions of the opening were read by the extensometer in the same way as the extension had been found. Fig. X. shows the result. Clearly the HOOKE's limit is again a well-marked point, and the calculated compressive stress is between 27,000 and 28,000 lbs. This agrees quite well with the value 29,000 found for the tensile limit

* 'History of Elasticity,' Vol. I., p. 887 *et seq.*

of the same hook. Accordingly it would seem that the proportionality of stress and strain in this material is reached at practically the same stretch and squeeze. Further, the method of determining this limit, whether by tension, or flexure, or

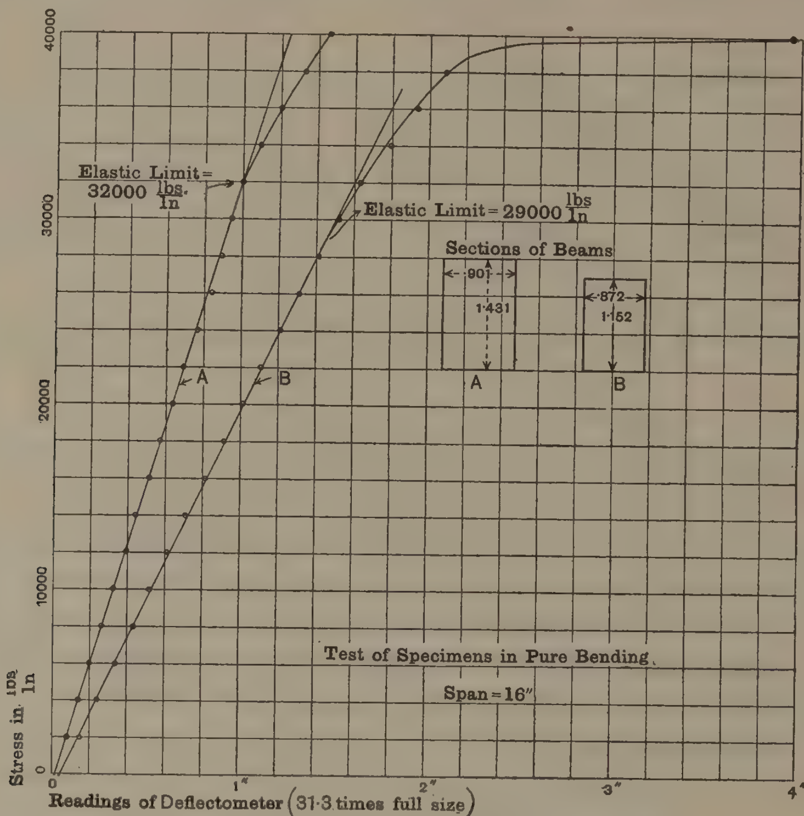


Fig. XIII.

bending of a hook, is indifferent, all processes leading to sensibly the same result. Therefore, at any rate for the steel of the present experiments, it seems the proper practical limit for all types of structure and forms of loading.

(8.) *General Conclusions.*

(i.) The existing theory of hooks is unsatisfactory both from the theoretical and experimental standpoints. It leads to very irregular results and precludes any chance of improving the design for hooks.

(ii.) The theory developed in this memoir allows for a number of factors omitted in the current theory, notably for the curvature of the belly and the change in size of the principal cross-section.

(iii.) It indicates that improvements can well be made in existing types and suggests the lines they should follow.

(iv.) It gives concordant results for hooks of different dimensions especially prepared, and also for a railway wagon coupling hook to which it was applied. In these cases the ordinary theory failed to express the facts.

(v.) The limit taken to compare the strength of hooks of various sizes was the limit to proportionality of stress and strain. If this be adopted as the limit to safe loading, the tensile stress at this limit is found to be the same whether we use a pure tensile test, a flexure test, or a hook test to determine it, and accordingly it is much to be preferred to the rough practical methods which note only the drop at the yield-point, and speak of this as the elastic limit.

(vi.) It is true that hooks are highly "worked" individual products; still we believe that series of crane or coupling hooks of the same material, but of different sizes, would give fairly homogeneous results, if tested with an extensometer for their HOOKE'S limits and if these limits were compared with the same limit reached on a tensile specimen of the same material.

We must heartily thank Professor CORMACK for allowing the use of his laboratory apparatus for the tests, and for help and suggestion at several stages.